Section 6: Sequences and proof

Solutions to Exercise

1. If n is odd, n^3 and n^2 are both odd, so $n^3 - n^2$ is even.
   If n is even, n^3 and n^2 are both even, so $n^3 - n^2$ is even.
   Therefore for all positive integers n, $n^3 - n^2$ is always even.

2. For three consecutive integers, at least one will be even, and one will be a multiple of 3. Therefore the product of the three integers is both a multiple of 2 and a multiple of 3, and so is a multiple of 6.

3. Any odd number can be written as $2n + 1$, where n is an integer.
   So the square of an odd number can be written as $(2n+1)^2 = 4n^2 + 4n + 1$
   $= 4n(n+1) + 1$
   One of n and $n + 1$ is even, so $n(n+1)$ is a multiple of 2 and therefore $4n(n+1)$ is a multiple of 8.
   So $4n(n+1)+1$ is one more than a multiple of 8. So the square of any odd number is always 1 more than a multiple of 8.

4. (i) nth term $= 3n - 1$
   1st term $= 3\times 1 - 1 = 2$
   2nd term $= 3\times 2 - 1 = 5$
   3rd term $= 3\times 3 - 1 = 8$
   4th term $= 3\times 4 - 1 = 11$
   Sequence is 2, 5, 8, 11, ...

   (ii) nth term $= n^2 - 1$
   1st term $= 1^2 - 1 = 0$
   2nd term $= 2^2 - 1 = 4 - 1 = 3$
   3rd term $= 3^2 - 1 = 9 - 1 = 8$
   4th term $= 4^2 - 1 = 16 - 1 = 15$
   Sequence is 0, 3, 8, 18, ...

   (iii) nth term $= 3n^2 - 2n + 1$
   1st term $= 3\times 1^2 - 2\times 1 + 1 = 3 - 2 + 1 = 2$
   2nd term $= 3\times 2^2 - 2\times 2 + 1 = 12 - 4 + 1 = 9$
   3rd term $= 3\times 3^2 - 2\times 3 + 1 = 27 - 6 + 1 = 22$
   4th term $= 3\times 4^2 - 2\times 4 + 1 = 48 - 8 + 1 = 41$
   Sequence is 2, 9, 22, 41, ...

5. (i) Each term increases by 3, so the general term must involve 3n.
nth term $= 3n - 1$.

(ii) Each term decreases by 2, so the general term must involve -2n.
nth term $= 12 - 2n$

6. (i) The sequence has nth term $an^2 + bn + c$.

Terms	3		9		17		27		39
Differences		6		8		10		12	
Second differences			2		2		2		

So $a = 1$

Terms	3	9	17	27	39
an^2	1	4	9	16	25
$bn + c$	2	5	8	11	14

The values of $bn + c$ go up by 3 each time, so $b = 3$, and $c = -1$

The nth term of the sequence is $n^2 + 3n - 1$.

(ii) The sequence has nth term $an^2 + bn + c$.

Terms	-2		4		14		28		46
Differences		6		10		14		18	
Second differences			4		4		4		

So $a = 2$

Terms	-2	4	14	28	46
an^2	2	8	18	32	50
$bn + c$	-4	-4	-4	-4	-4

The values of $bn + c$ are all the same, so $b = 0$, and $c = -4$

The nth term of the sequence is $2n^2 - 4$.

(iii) The sequence has nth term $an^2 + bn + c$.

Terms	7		12		15		16		15
Differences		5		3		1		-1	
Second differences			-2		-2		-2		

So $a = -1$

Terms	7	12	15	16	15
an^2	-1	-4	-9	-16	-25
$bn + c$	8	16	24	32	40

The values of $bn + c$ go up by 8 each time, so $b = 8$, and $c = 0$

The nth term of the sequence is $-n^2 + 8n$.

7. (i) $\text{nth term} = \dfrac{2n+5}{4n-1}$

$1^{st}\text{ term} = \dfrac{2\times1+5}{4\times1-1} = \dfrac{2+5}{4-1} = \dfrac{7}{3}$

$5^{th}\text{ term} = \dfrac{2\times5+5}{4\times5-1} = \dfrac{10+5}{20-1} = \dfrac{15}{19}$

$100^{th}\text{ term} = \dfrac{2\times100+5}{4\times100-1} = \dfrac{200+5}{400-1} = \dfrac{205}{399}$

As $n\to\infty$, $2n+5\to 2n$, $4n-1\to 4n$

$$\text{so } \frac{2n+5}{4n-1} \to \frac{2n}{4n} = \frac{1}{2}$$

The limit of the sequence is $\frac{1}{2}$.

(ii) $\text{nth term} = \dfrac{1-6n}{2n+3}$

$1^{st}\text{ term} = \dfrac{1-6\times1}{2\times1+3} = \dfrac{1-6}{2+3} = -\dfrac{5}{5} = -1$

$5^{th}\text{ term} = \dfrac{1-6\times5}{2\times5+3} = \dfrac{1-30}{10+3} = -\dfrac{29}{13}$

$100^{th}\text{ term} = \dfrac{1-6\times100}{2\times100+3} = \dfrac{1-600}{200+3} = -\dfrac{599}{203}$

As $n\to\infty$, $1-6n\to -6n$, $2n+3\to 2n$

$$\text{so } \frac{1-6n}{2n+3} \to \frac{-6n}{2n} = -3$$

8. <u>Method 1</u>

$n^2+2n-5=1000$

$n^2+2n=1005$

$n^2+2n+1=1006$

$(n+1)^2=1006$

$n+1$ is a whole number but 1006 is not a square number so 1000 cannot be a term in the sequence.

<u>Method 2</u>

$n^2 + 2n - 5 = 1000$

$n^2 + 2n = 1005$

$n(n+2) = 1005$

n and $n+2$ are consecutive even or consecutive odd numbers. To multiply to make 1005, they must both be odd.

$1005 = 3 \times 5 \times 67$

There are no consecutive odd numbers that multiply to make 1005 so 1000 cannot be a term in the sequence.

Method 3

$n^2 + 2n - 5 = 1000$

$n^2 + 2n - 1005 = 0$

Solve the quadratic equation.

$n = 30.72$ or -32.72

n is not an integer so 1000 is not a term in the sequence.

9. (a) One possible sequence is $\dfrac{3n}{n+1}$

(b) One possible sequence is $4 - \dfrac{n}{n+1}$