Section 3: Trig graphs, identities and equations

Notes and Examples

In this section you learn how to solve trigonometric equations.

These notes contain subsections on

- Trigonometric identities
- Principal values
- Solving simple trigonometrical equations
- More complicated examples of trigonometrical equations.

Trigonometric identities

You need to learn the following identities:

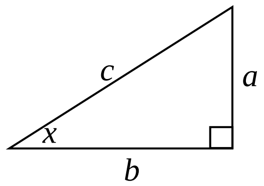
$$\tan x \equiv \frac{\sin x}{\cos x}$$

e.g. $\tan 30° = \frac{\sin 30°}{\cos 30°}$

$$\sin^2 x + \cos^2 x \equiv 1$$

An identity is true for all values of x.

You can prove the identities quite easily using a right-angled triangle.



$$\frac{\sin x}{\cos x} = \frac{a}{c} \div \frac{b}{c} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b} = \tan x$$

$$\sin^2 x + \cos^2 x = \frac{a^2}{b^2} + \frac{b^2}{c^2} = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$$

Using Pythagoras' theorem

In the next example you need to use the trigonometric identities to rewrite an expression.

Example 1

Show that $(\sin\theta + \cos\theta)(\sin\theta - \cos\theta) = 2\sin^2\theta - 1$

Solution

Working with the LHS and expanding the brackets gives:

Since $\sin^2\theta + \cos^2\theta \equiv 1$ then $\cos^2\theta \equiv 1 - \sin^2\theta$ ②

Substituting ② into ① gives:

$$(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)=\sin^2\theta-(1-\sin^2\theta)$$

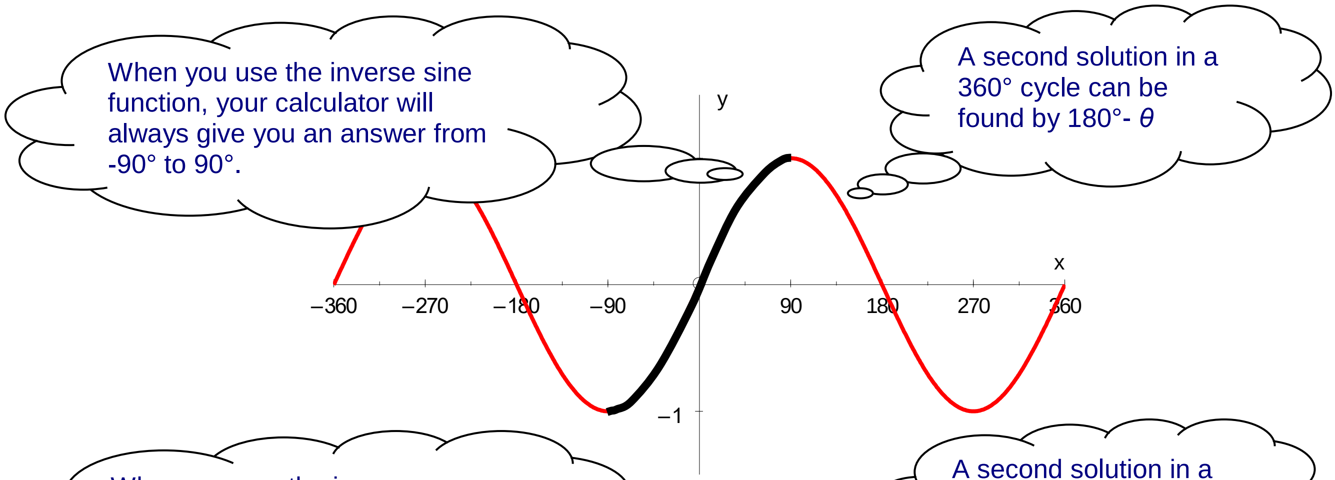
Simplifying: $(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)=2\sin^2\theta-1$ as required.

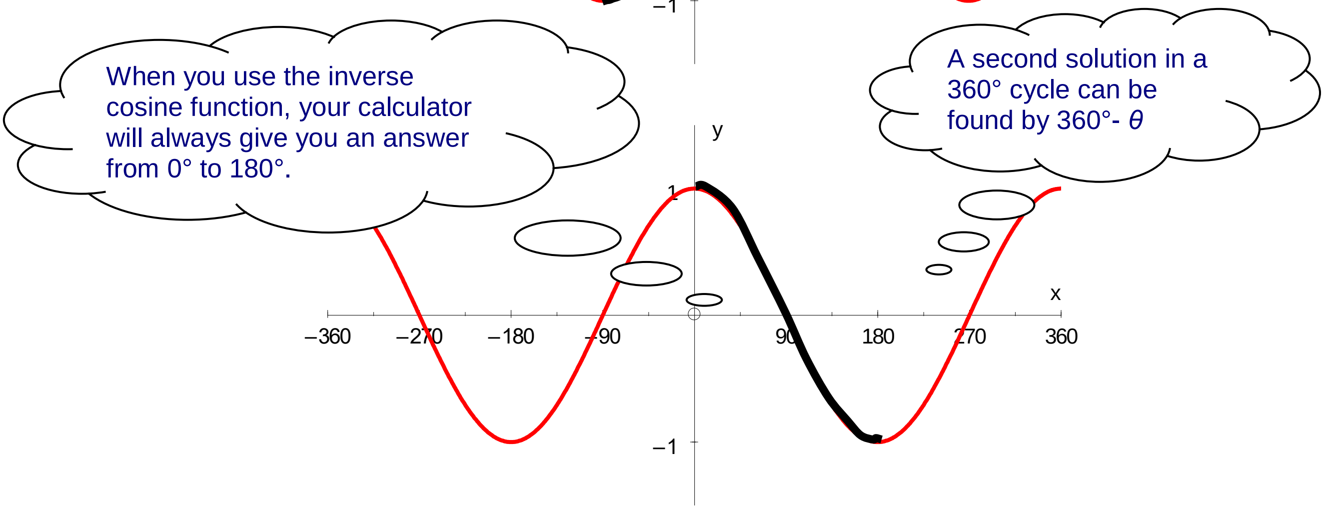
Principal values

There are infinitely many roots to an equation like $\sin\theta=\frac{1}{2}$.

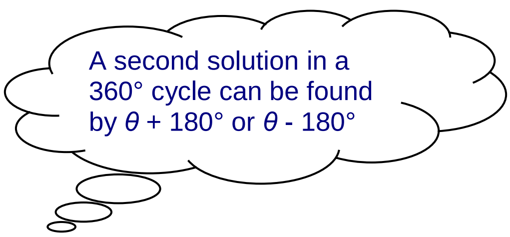
Your calculator will only give one solution – the ***principal value***.
You find this by pressing the calculator keys for $\sin^{-1}$ 0.5 (or arcsin 0.5 or invsin 0.5). Check that you can get the answer of 30°.

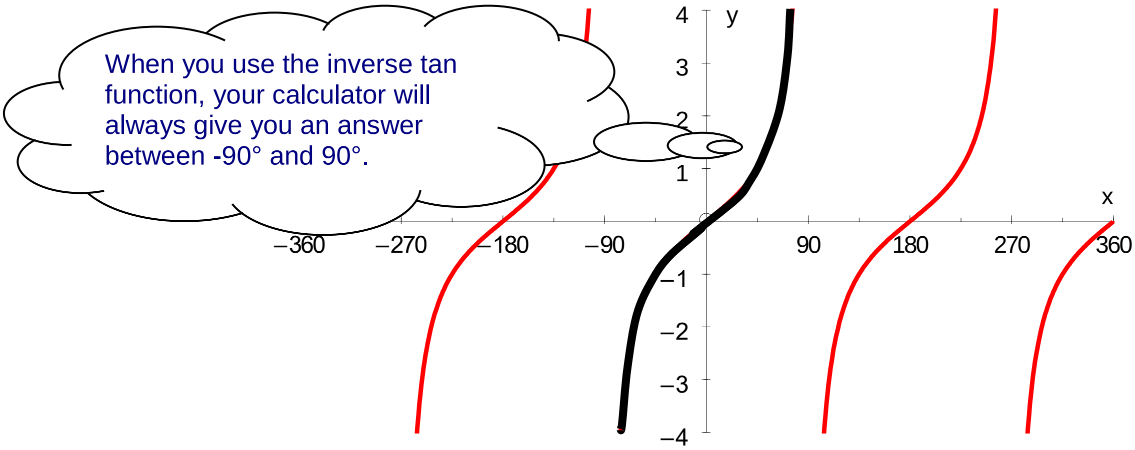
You can find other roots by looking at the symmetry of the appropriate graph.





$y = \tan\theta$

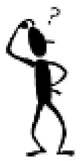




Alternatively, you can use the quadrant diagram to find other solutions, by thinking about which quadrants the solutions will be in.

Solving simple trigonometrical equations

Because there are infinitely many solutions to a trigonometric equation you are only ever asked to find a few of them! Any question at this level asking you to solve a trigonometric equation will also give you the interval or range of values in which the solutions must lie, e.g. you might be asked to solve $\tan\theta = 2$ for $0° \le \theta \le 360°$.
You can only directly solve trigonometric equations like $\sin\theta = \frac{1}{2}$ or $\cos\theta = \frac{1}{4}$ or $\tan\theta = -2$. Here is an example.



Example 2
Solve the equations

(i) $\cos x = \dfrac{\sqrt{3}}{2}$ for $0° \le x \le 360°$.

(ii) $\sin x = -0.2$ for $0° \le x \le 360°$

Solution
(i) cos is positive in the 1st and 4th quadrants.

$\cos x = \dfrac{\sqrt{3}}{2} \Rightarrow x = 30°$

There will be a second solution in the 4th quadrant.
360° - 30° = 330° is also a solution.

So the values of x for which $\cos x = \dfrac{\sqrt{3}}{2}$ are 60° and 120°.

(ii) sin is negative in the 3rd quadrant and the 4th quadrant
Using a calculator, $\sin x = -0.2 \Rightarrow x = -11.53°$
This is not in the required range.

The solution in the 3rd quadrant is 180° + 11.53° = 191.53°.
The solution in the 4th quadrant is 360° - 11.53° = 348.47°
So the values of x for which $\sin x = -0.2$ are 191.53° and 348.47° (2 d.p.)

More complicated trigonometrical equations

Any more complicated equations need to be manipulated algebraically before they can be solved. There are a number of techniques you can use:

1. Rearrange the equation to make $\cos\theta$, $\sin\theta$ or $\tan\theta$ the subject.
2. Check to see if the equation factorises to give two (or more) equations which involve just one trigonometric function (see Example 3).
   If it is a quadratic in either $\sin\theta$, $\cos\theta$, or $\tan\theta$ it can either be factorised or solved using the formula for solving quadratic equations (see Example 4).
3. If the equation involves just $\sin\theta$ and $\cos\theta$ (and no powers), check to see if you can use the identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ (see Example 5).
4. If the equation contains a mixture of trigonometric functions (e.g. $\cos^2\theta$ and $\sin\theta$) then you may need to use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to make it a quadratic in either $\sin\theta$, $\cos\theta$, or $\tan\theta$ (see Example 6).

Example 3
Solve $2\cos\theta\sin\theta + \cos\theta = 0$ for $0° \le \theta \le 360°$.

Solution
$2\cos\theta\sin\theta + \cos\theta = 0$ can be factorised as there is $\cos\theta$ in both terms on the LHS.
Factorise: $\cos\theta(2\sin\theta + 1) = 0$
So either $\cos\theta = 0$ or $2\sin\theta + 1 = 0$

It is wrong to divide through by $\cos\theta$ because you lose the solutions to $\cos\theta = 0$.

$\cos\theta = 0 \Rightarrow \theta = 90°$
360° - 90° = 270° is also a solution.

$2\sin\theta + 1 = 0 \Rightarrow \sin\theta = -\tfrac{1}{2}$
This has solutions in the third and fourth quadrants.
The solutions are 180° + 30° = 210° and 360° - 30° = 330°.

So the values of θ for which $2\cos\theta\sin\theta + \cos\theta = 0$ are 90°, 210°, 270° and 330°.

In Example 4 you need to solve a quadratic equation.

Example 4
Solve $2\cos^2\theta + 3\cos\theta = 2$ for $0° \le \theta \le 360°$.

Solution

You can replace $\cos\theta$ with x to make things simpler! Or factorise straightaway to get: $(2\cos\theta - 1)(\cos\theta + 2) = 0$ and then solve.

$2\cos^2\theta + 3\cos\theta = 2$ is a quadratic equation in $\cos\theta$

Rearrange the quadratic: $2\cos^2\theta + 3\cos\theta - 2 = 0$

Let $\cos\theta = x$: $2x^2 + 3x - 2 = 0$

Factorise: $(2x-1)(x+2) = 0$

$x = \frac{1}{2}$ or $x = -2 \Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -2$

$\cos\theta = -2$ has no solutions.

So we need to solve $\cos\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = 60°$

There is also a solution in the 4th quadrant, so 360° - 60° = 300° is also a solution.

So the values of θ for which $2\cos^2\theta + 3\cos\theta = 2$ are 60° and 300°.

In the next example you need to use the identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$.

Example 5

Solve $\sin\theta - 2\cos\theta = 0$ for $0° \le \theta \le 360°$.

Solution

You need to rearrange the equation.

$\sin\theta - 2\cos\theta = 0$

Dividing by cos θ: $\dfrac{\sin\theta}{\cos\theta} - 2 = 0$

Since $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$: $\tan\theta - 2 = 0$

$\Rightarrow \tan\theta = 2$

$\Rightarrow \theta = 63.4°$ to 1 d.p.

You can safely divide by cos θ because it can't be equal to 0. If it were then sin θ would also have to be 0 and cos θ and sin θ are never both 0 for the same value of θ.

There is also a solution in the 3rd quadrant.

So 63.4° + 180° = 243.4° is also a solution.

So the values of θ for which $\sin\theta - 2\cos\theta = 0$ are 63.4° and 243.4° to 1 d.p.

In the next example you need to use the trigonometric identity $\sin^2\theta + \cos^2\theta \equiv 1$.

Example 6

Solve $\sin^2 x + \sin x = \cos^2 x$ for $0° \le x \le 360°$

Solution

Rearranging the identity $\sin^2\theta + \cos^2\theta \equiv 1$

gives: $\cos^2 x \equiv 1 - \sin^2 x$ ①

Substituting ① into the equation $\sin^2 x + \sin x = \cos^2 x$ gives:

$$\sin^2 x + \sin x = 1 - \sin^2 x$$

This is a quadratic in $\sin x$.

Rearranging: $2\sin^2 x + \sin x - 1 = 0$

Rearranging: $2\sin^2 x + \sin x - 1 = 0$

This factorises to give: $(2\sin x - 1)(\sin x + 1) = 0$

So either: $2\sin x - 1 = 0$ or $\sin x + 1 = 0$

$\Rightarrow \sin x = \frac{1}{2}$ $\qquad$ $\Rightarrow \sin x = -1$

$\Rightarrow x = 30°$ or $150°$ $\qquad$ $\Rightarrow x = 270°$

So the solutions to $\sin^2 x + \sin x = \cos^2 x$ are $x = 30°, 150°$ or $270°$