Section 4: Linear and quadratic inequalities

Notes and Examples

These notes contain subsections on

Inequalities

Inequalities are similar to equations, but instead of an equals sign, =, they involve one of these signs:

$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

This means that whereas the solution of an equation is a specific value, or two or more specific values, the solution of an inequality is a range of values.

Inequalities can be solved in a similar way to equations, but you do have to be very careful, as in some situations you need to reverse the inequality. This is shown in these examples.

Linear inequalities

A linear inequality involves only terms in x and constant terms.

Example 1
Solve the inequality $3x+1>x-5$

You can treat this just like a linear equation.

Solution

$3x+1>x-5$

$2x+1>-5$ (Subtract x from each side)

$2x>-6$ (Subtract 1 from each side)

$x>-3$ (Divide both sides by 2)

The next example involves a situation where you have to divide by a negative number. When you are solving an equation, multiplying or dividing by a negative number is not a problem. However, things are different with inequalities.
The statement $-3<2$ is clearly true.

If you add something to each side, it is still true

If you subtract something from each side, it is still true

$-3 + 1 < 2 + 1$
$-2 < 3$ ✓

$-3 - 4 < 2 - 4$
$-7 < -2$ ✓

If you multiply or divide each side by a positive number, it is still true

$-3 \times 2 < 2 \times 2$
$-6 < 4$ ✓

However, if you multiply each side by a negative number then things go wrong!

$-3 \times -2 < 2 \times -2$
$6 < -4$ ✗

When you multiply or divide each side by a negative number, you must reverse the inequality.

The following example demonstrates this. Two solutions are given: in the first the inequality is reversed when dividing by a negative number, in the second this situation is avoided by a different approach.

Example 2
Solve the inequality $1 - x \geq 2x - 5$

Solution (1)

$$1 - x \geq 2x - 5$$
Subtract $2x$ from each side
$$1 - 3x \geq -5$$
Subtract 1 from each side
$$-3x \geq -6$$
Divide both sides by -3, reversing the inequality.
$$x \leq 2$$

Solution (2)

$$1 - x \geq 2x - 5$$
Add x to each side
$$1 \geq 3x - 5$$
Add 5 to each side
$$6 \geq 3x$$
Divide both sides by 3
$$2 \geq x$$
Finish by writing the inequality the other way round.
$$x \leq 2$$

You can check that you have the sign th... ...in the range of the solution, and checking tha... ...In the above example, you could try $x = 1$. In the original inequality you get $0 \geq -3$, which is correct.

Quadratic inequalities

You can solve a quadratic inequality by factorising the quadratic expression, just as you do to solve a quadratic equation. This tells you the boundaries of the solutions. The easiest way to find the solution is then to sketch a graph.


Example 3

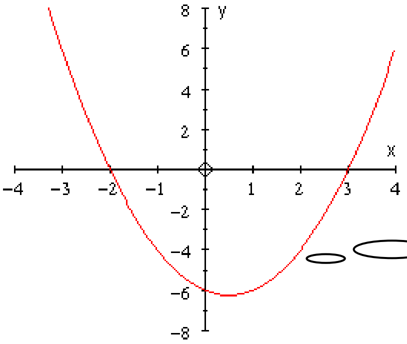
Solve the inequality $x^2 - x - 6 < 0$

Solution

$x^2 - x - 6 < 0$

$(x+2)(x-3) < 0$

This shows that the graph of $y = x^2 - x - 6$ cuts the x-axis at x = -2 and x = 3. Use this information to sketch the graph.



The solution to the inequality is the negative part of the graph. This is the part between –2 and 3.

The solution is $-2 < x < 3$


Example 4

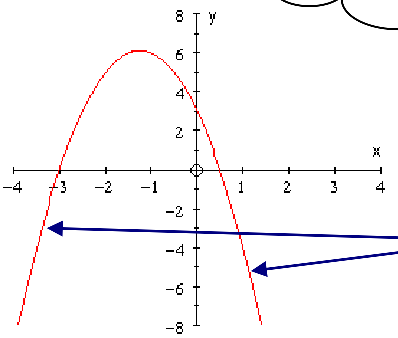
Solve the inequality $3 - 5x - 2x^2 \le 0$

Solution

$3 - 5x - 2x^2 \le 0$

$(3+x)(1-2x) \le 0$

This shows that the graph of $y = 3 - 5x - 2x^2$ cuts the x-axis at x = -3 and x = ½. You can now sketch the graph – note that as the term in x^2 is negative, the graph is inverted.



The solution to the inequality is the negative part of the graph. This is in fact two separate parts.

The solution is $x \le -3$ or $x \ge ½$.

Note: if you prefer to work with a positive x^2 term, you can change all the signs in the original inequality and reverse the inequality, giving $2x^2 + 5x - 3 \ge 0$. The graph will then be the other way up, and you will take the positive part of the graph, so the solution will be the same.

